Acceleration of quantum fields

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Abstract

We analyze the transformation of quantum fields under conformal coordinate transformations from inertial to accelerated frames, in the simple case of scalar massless fields in a two-dimensional spacetime, through the transformation of particle number and its spectral density. Particle number is found to be invariant under conformal coordinate transformations to uniformly accelerated frames, which extends the property already known for vacuum. Transformation of spectral density of particle number exhibits a redistribution of particles in the frequency spectrum. This redistribution is determined by derivatives of phase operators with respect to frequency, that is by time and position operators defined in such a manner that the redistribution of particles appears as a Doppler shift which depends on position in spacetime, in conformity with Einstein equivalence principle.

PACS numbers: 12.20 Ds 03.70 42.50 Lc

LPTENS 95/30
to appear in Brazilian Journal of Physics,
special issue on Quantum Optics, ed. L. Davidovich.

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1 Introduction

Lorentz invariance of electromagnetism lies at the heart of the theory of relativity[1]. This is true not only for the classical theory of electromagnetism, but also for the quantum theory. In particular, invariance of vacuum fluctuations under Lorentz transformations is needed to ensure that mechanical effects of these fluctuations preserve the relativity of uniform motion in empty space.

In contrast with these universally accepted ideas, the interplay between quantum fields and accelerated frames has been the object of much debate. Since Einstein[2], accelerated frames are commonly represented by using Rindler changes of coordinates[3] between inertial and accelerated frames. These transformations do not preserve the propagation equations of electromagnetic fields. Light rays appear curved in accelerated frames while frequencies undergo a shift during light propagation[2]. Such a representation of accelerated frames also results in a transformation of vacuum into a thermal bath[4]. This idea has apparently been easily accepted because of its association with the most spectacular effect predicted by quantum field theory in curved spacetime, namely thermal particle creation due to curvature[5]. It is nevertheless clear that accelerated frames and curved spacetime are completely different physical problems, from the point of view of general relativity. Furthermore, the notion of particle number plays a central role in the interpretation of quantum field theory, and the fact that it is not preserved in accelerated frames leads to weighty paradoxes for quantum theory[6]. These difficulties also raise doubts about the significance of the Einstein equivalence principle in the quantum domain. Such a principle indeed relies on the very notion of particle number, as well as on the interpretation of frequency change from inertial to accelerated frames as a Doppler shift depending on position in spacetime[2]. If vacuum or 1-photon states may not be defined in a consistent manner in inertial and accelerated frames, it might appear hopeless to attribute any significance to such a principle in the quantum domain.

In the present paper, we show that the interplay between acceleration and quantum fields may be analyzed in a consistent manner which allows to extend the Einstein equivalence principle to the quantum domain. Our approach makes use of the conformal symmetry of quantum theory of massless fields, like a scalar field in two-dimensional (2D) spacetime or the electromagnetic field in four-dimensional (4D) spacetime. For the sake of simplicity, we will restrict our attention here to the 2D case.

It has been known for a long time that the symmetry related to inertial motions, associated with Lorentz transformations, can be extended for massless fields to a larger group which includes conformal transformations to accelerated frames[7]. Light rays remain straight lines with such a representation of accelerated frames while frequencies are now preserved during

light propagation. These transformations are known to fit[8] the relativistic definition of uniformly accelerated motion[9]. It has also been shown that vacuum remains unchanged under conformal transformations to accelerated frames[10]. Here, we extend the latter property by demonstrating the invariance of total particle number under such transformations. This proves the consistency of a point of view which maintains invariance of vacuum and particle number for inertial and accelerated observers.

When a spectral decomposition of particle number is introduced and the transformation of spectral density from inertial to accelerated frames analyzed, field phase operators make an appearance. We will show that the resulting expressions correspond to quantum definitions for position in spacetime[11] which comply with the requirement of the Einstein equivalence principle for the interpretation of acceleration on quantum fields in terms of Doppler shifts. It is well known that the definition of phase operators, which may be considered as conjugated to the number operators[12], leads to ambiguities[13, 14]. A lot of work has been devoted to cure these ambiguities (see[15] and references therein). However, the conclusions that we will reach in the present paper will essentially be unaffected by these difficulties.

2 Conformal coordinate transformations

In a two-dimensional (2D) spacetime, a free massless scalar field $\phi(t,x)$ is the sum of two counterpropagating components:

$$\phi(t,x) = \varphi(u) + \psi(v) \qquad u = t - x \qquad v = t + x \tag{1}$$

We use natural spacetime units ($c = 1$); t is the time coordinate, x is the space coordinate, u and v are the two light-cone variables.

In the 2D case, conformal coordinate transformations are those transformations which act separately on the two light-cone variables, and they are specified by arbitrary functions f and g describing the relations between such variables in the two reference systems:

$$\overline{u} = f(u) \qquad \overline{v} = g(v) \tag{2}$$

The field transformation under conformal coordinate transformations is defined through:

$$\varphi \to \overline{\varphi} \qquad \varphi(u) = \overline{\varphi}(\overline{u}) \tag{3}$$

From now on, we consider only one (φ) of the two counterpropagating components; the other one (ψ) can be dealt with in exactly the same way.

As well-known in Quantum Field Theory, the field transformation under coordinate transformations may be considered as generated by linear forms of the stress tensor, that is also quadratic forms of the fields[16]. In order to give

explicit forms of these generators in the spectral domain, we introduce the Fourier components $\varphi[\omega]$ of the field $\varphi(u)$ according to the general definition:

$$\varphi(u) = \int \frac{d\omega}{2\pi} \varphi[\omega] e^{-i\omega u} \tag{4}$$

These components are related to the standard annihilation and creation operators:

$$\begin{aligned}
\varphi[\omega] &= \sqrt{\frac{\hbar}{2|\omega|}} \left(\theta(\omega) a_\omega + \theta(-\omega) a^\dagger_{-\omega}\right) && (5)\\
[a_\omega, a_{\omega'}] &= \left[a^\dagger_\omega, a^\dagger_{\omega'}\right] = 0 && (6)\\
\left[a_\omega, a^\dagger_{\omega'}\right] &= 2\pi\delta(\omega - \omega') && (7)
\end{aligned}$$

where θ is the Heaviside function and δ the Dirac distribution. The commutation relations of the Fourier components of the field are given by:

$$[\varphi[\omega], \varphi[\omega']] = \frac{\pi\hbar}{\omega} \delta(\omega + \omega') \tag{8}$$

We now define the generating function $T[\omega]$:

$$T[\omega] = \int \frac{d\omega'}{2\pi} \omega'(\omega + \omega') \varphi[-\omega'] \varphi[\omega + \omega'] \tag{9}$$

as the Fourier transform of the stress tensor[3]:

$$T(u) = (\partial_u \varphi(u))^2 \tag{10}$$

We then introduce the generators T_k as the coefficients of the Taylor expansion of the generating function (k a positive integer):

$$T_k = \left\{(-i\partial_\omega)^k T[\omega]\right\}_{\omega=0} = \int u^k T(u)\, du \tag{11}$$

The commutators of these quantities with the field are obtained as:

$$\begin{aligned}
[T[\omega], \varphi[\omega']] &= -\hbar(\omega + \omega') \varphi[\omega + \omega'] && (12)\\
[T_k, \varphi[\omega]] &= -\hbar(-i\partial_\omega)^k \{\omega\varphi[\omega]\} && (13)
\end{aligned}$$

The latter relation precisely fits the action upon the field of an infinitesimal conformal coordinate transformation. Denoting:

$$\delta\varphi[\omega] = \frac{\varepsilon}{i\hbar} [T_k, \varphi[\omega]] \tag{14}$$

[3]In this definition appear symmetric products of the field operators, rather than normally ordered products. As a consequence, the generating function describes the stress tensor associated with vacuum, as well as the stress tensor associated with particles.

with ε an infinitesimal real number, one indeed deduces:

$$\delta\varphi(u) = -\varepsilon u^k \partial_u \varphi(u) \tag{15}$$

This corresponds to equations (2-3) with an infinitesimal coordinate transformation:

$$\begin{aligned} \overline{u} &= u + \delta f(u) && (16) \\ \delta\varphi(u) &\equiv \overline{\varphi} - \varphi = -\delta f(u)\, \partial_u \varphi(u) && (17) \\ \delta f(u) &= \varepsilon u^k && (18) \end{aligned}$$

Notice that the generating function $T[\omega]$ may also be associated with an infinitesimal coordinate transformation:

$$\delta f(u) = \varepsilon \exp(i\omega u) \tag{19}$$

This does not correspond to a real coordinate tranformation which would necessarily involve opposite values of the frequency[4].

In order to recover the known commutation relations for the conformal generators[17, 18], we write the commutator of the generating function with quadratic forms of the field:

$$\begin{aligned} \left[T[\omega], \varphi[\omega']\varphi[\omega'']\right] &= -\hbar\{(\omega+\omega')\,\varphi[\omega+\omega']\,\varphi[\omega''] \\ &\quad + (\omega+\omega'')\,\varphi[\omega']\,\varphi[\omega+\omega'']\} && (20) \end{aligned}$$

We then deduce the commutator of the generating function evaluated at different arguments:

$$\left[T[\omega], T[\omega']\right] = \hbar(\omega - \omega')\, T(\omega + \omega') \tag{21}$$

A Taylor expansion of this relation provides commutators characteristic of the conformal algebra (for positive integers k)[5]:

$$\left[T_k, T_{k'}\right] = i\hbar\,(k' - k)\, T_{k+k'-1} \tag{22}$$

[4]We may emphasize that we are dealing with coordinate transformations of real Minkowski spacetime, here represented by real light-cone variables. The generating function (9), that is the Fourier transform of the field stress tensor, is not hermitian but satisfies $T[\omega]^\dagger = T[-\omega]$. The generators T_k are defined in equation (11) as hermitian operators, in contrast with common definitions of generators in Conformal Field Theory in 2D spacetime[18].

[5]Note that the original Virasoro algebra is defined with negative and positive order non-hermitian generators. It is generated by $\frac{1}{\hbar\Omega}T[k\Omega]$, where Ω is a scale frequency. It also corresponds to a Laurent expansion in variable u of the function $T(u)$ extended to the complex plane[18].

3 Transformation of vacuum

The conformal coordinate transformations preserve the propagation equation of massless fields, and therefore their commutators[10]. However, not all of them preserve vacuum fluctuations.

The vacuum state is defined by specific correlation functions:

$$\langle \varphi[\omega]\,\varphi[\omega']\rangle_{\mathrm{vac}} = \theta(\omega)\,\theta(-\omega')\,[\varphi[\omega]\,,\varphi[\omega']] \tag{23}$$

$\langle\ \rangle_{\mathrm{vac}}$ represents a mean value in the vacuum state. This means that annihilators vanish when applied to the vacuum state. Using expression (8) of the field commutators, one obtains the correlation function:

$$\langle \varphi[\omega]\,\varphi[\omega']\rangle_{\mathrm{vac}} = \theta(\omega)\,\frac{\pi\hbar}{\omega}\delta(\omega+\omega') \tag{24}$$

Using transformation (20) of field quadratic forms, one then deduces the transformation of vacuum correlation functions:

$$\langle [T[\omega]\,,\varphi[\omega']\,\varphi[\omega'']]\rangle_{\mathrm{vac}} = \pi\hbar^2\,(\theta(\omega')-\theta(-\omega''))\,\delta(\omega+\omega'+\omega'') \tag{25}$$

It is also worth writing the transformation of the vacuum stress tensor, that is of the generating function itself[6]:

$$\langle [T[\omega]\,,T[\omega']]\rangle_{\mathrm{vac}} = \frac{\hbar^2\omega^3}{12}\delta(\omega+\omega') \tag{26}$$

One then demonstrates, through a Taylor expansion of these relations in the frequency ω, that the vacuum correlation function for field derivatives $\partial_u\varphi$ (which correspond to Fourier components $-i\omega\varphi[\omega]$), as well as the vacuum stress tensor, are preserved by the infinitesimal generators T_0, T_1 and T_2 which respectively describe translations, Lorentz boosts and conformal transformations from inertial to accelerated frames. This is no longer the case for the higher-order generators. In particular, the generator T_3 changes the vacuum stress tensor in a manner which is consistent with the dissipative force felt by a mirror moving in vacuum with a non-uniform acceleration[19, 20].

We thus recover the result of reference[10] for a massless scalar field theory in a 2D spacetime: the vacuum is not invariant under the large group of conformal coordinate transformations (equation (2) with an arbitrary function f). It is invariant only under the smaller group of transformations generated by T_0, T_1 and T_2. Those transformations correspond to the particular case of homographic functions[7]:

$$\overline{u} = \frac{au+b}{cu+d} \qquad\qquad ad-bc=1 \tag{27}$$

[6]When written in terms of normally ordered products, commutation relations (21) between the generators include a further pure number. This central charge is determined by equation (26).

[7]Note that the modification of the mean vacuum stress tensor $\langle T(u)\rangle_{\mathrm{vac}}$ under a conformal transformation associated with the function f is proportional to the Schwartzian derivative of f, which vanishes for homographic transformations[18, 19].

In the following, we give some results for the large conformal group, but we focus our attention onto the smaller group of transformations which preserve vacuum, and particularly onto the action of the acceleration generator T_2.

4 Transformation of particle number operators

We will denote n_ω the spectral density of particle number:

$$n_\omega = a^\dagger_\omega a_\omega = \frac{2\omega}{\hbar}\theta(\omega)\varphi[-\omega]\varphi[\omega] \tag{28}$$

The values at different frequencies are commuting quantities:

$$[n_\omega, n_{\omega'}] = 0 \tag{29}$$

and the commutators with the field may be written from relations (8):

$$[n_\omega, \varphi[\omega']] = 2\pi\{\delta(\omega+\omega') - \delta(\omega-\omega')\}\varphi[\omega'] \tag{30}$$

This definition is such that the generator T_0, that is the field energy, has its standard form in terms of number density:

$$T_0 = \langle T_0\rangle_{\text{vac}} + \int_0^\infty \frac{d\omega}{2\pi}\hbar\omega n_\omega \tag{31}$$

The total number n of particles is defined as the integral of n_ω:

$$n = \int_0^\infty \frac{d\omega}{2\pi} n_\omega \tag{32}$$

The number operators n_ω are defined for positive frequencies, and vanish when applied to the vacuum state.

We come now to the main argument of the present paper, that is the transformation of particle numbers under conformal coordinate transformations. As an immediate consequence of transformation (20) of field quadratic forms, we deduce the transformation of the number density:

$$\begin{aligned}[T[\omega], n_{\omega'}] &= -2\omega'\theta(\omega')\{(\omega-\omega')\varphi[\omega-\omega']\varphi[\omega'] \\ &\quad + (\omega+\omega')\varphi[-\omega']\varphi[\omega+\omega']\}\end{aligned} \tag{33}$$

We obtain the transformation of the total particle number by an integration:

$$[T[\omega], n] = -\int_0^\omega \frac{d\omega'}{\pi}\omega'(\omega-\omega')\varphi[\omega-\omega']\varphi[\omega'] \tag{34}$$

We then derive the effect of the infinitesimal generators by performing a Taylor expansion in the frequency ω of the previous expressions.

The total particle number n is preserved by the generators T_0, T_1 and T_2:

$$\frac{1}{i\hbar}\left[T_0, n\right] = \frac{1}{i\hbar}\left[T_1, n\right] = \frac{1}{i\hbar}\left[T_2, n\right] = 0 \tag{35}$$

This property is well-known for the translations and Lorentz boosts. The new result is that a conformal transformation to an accelerated frame also leads to a redistribution of particles in the frequency domain, without any change of the total number of particles. It is consistent with the invariance of vacuum in the homographic group generated by T_0, T_1 and T_2, as discussed in the previous section. It means that the notion of particle number is the same for accelerated observers and for inertial ones, provided that accelerated frames are defined through conformal transformations. For the other generators $T_{k\geq 3}$, the vacuum is no longer preserved and the total particle number n is changed[8].

We now write the transformation of the spectral density n_ω of particle number under the generators T_0, T_1 and T_2 which perserve the total number n. As expected, the number density is unchanged under a translation:

$$\frac{1}{i\hbar}\left[T_0, n_\omega\right] = 0 \tag{36}$$

but changed under a Lorentz boost:

$$\frac{1}{i\hbar}\left[T_1, n_\omega\right] = \partial_\omega \left\{\omega n_\omega\right\} \tag{37}$$

This latter change is a mere mapping in the frequency domain, associated with the Doppler shift of the field frequency. We then write the modification of the spectral density of particle number in a conformal transformation from an inertial to an accelerated frame:

$$\frac{1}{i\hbar}\left[T_2, n_\omega\right] = 2\partial_\omega \left\{\omega m_\omega\right\} \tag{38}$$

$$m_\omega = \frac{\omega}{i\hbar}\theta\left(\omega\right)\left\{\varphi\left[-\omega\right]\varphi'\left[\omega\right] + \varphi'\left[-\omega\right]\varphi\left[\omega\right]\right\} \tag{39}$$

The quadratic form m_ω is hermitian. It may not be rewritten in terms of the density n_ω or its derivatives. In other words, the modification of n_ω under T_2 amounts to a redistribution of particles in the frequency domain, without any change of the total particle number, as it was the case for the modification of n_ω under T_1, but this redistribution is no longer equivalent to a mere mapping of the density n_ω in the frequency spectrum. We will show later on that the expression (39) may be interpreted as a Doppler shift which depends on position in spacetime, in conformity with Einstein equivalence principle.

[8]Note that the commutator (33) vanishes when applied to the vacuum state, for arbitrary positive frequencies ω. However, vacuum and particle numbers are not invariant under generators $T_{k\geq 3}$. These properties are consistent since, as already mentioned, $T\left[\omega\right]$ is not hermitian and real coordinate transformations involve the generating function $T\left[\omega\right]$ at negative frequencies as well as positive ones.

5 Quantum phase and phase-time operators

In the present section, we show how to obtain quantum operators associated with positions in spacetime.

As a first step in this direction, we introduce operators e_ω and δ_ω such that:

$$\begin{aligned} a_\omega &= e_\omega \sqrt{n_\omega} && (40) \\ a_\omega^\dagger &= \sqrt{n_\omega} e_\omega^\dagger && (41) \\ e_\omega &= e^{i\delta_\omega} && (42) \end{aligned}$$

As well-known, these relations are not sufficient to define phase operators since annihilators and creators are not modified by a redefinition of the phases such that:

$$e_\omega \rightarrow e_\omega + \pi_\omega \qquad\qquad \pi_\omega n_\omega = 0 \tag{43}$$

Various definitions of the phase operators, for example the Susskind-Glogower definition[14] or the Pegg-Barnett definition[21], are connected through such redefinitions. We show below that the properties studied in the present paper may be stated independently of such ambiguities.

We now list some properties which are satisfied for any operators defined from relations (40-42); these properties depend only upon the field commutation relations (6-7). First, the exponential operators e_ω are commuting variables, like the number operators (compare with (29)):

$$[e_\omega, e_{\omega'}] = 0 \tag{44}$$

This is also the case for their adjoint operators $e_\omega^\dagger$:

$$\left[e_\omega^\dagger, e_{\omega'}^\dagger\right] = 0 \tag{45}$$

The commutation relations between operators e_ω (or $e_\omega^\dagger$) and the number operators n_ω satisfy:

$$\begin{aligned} [n_\omega, e_{\omega'}] \sqrt{n_{\omega'}} &= -2\pi e_{\omega'} \delta(\omega - \omega') \sqrt{n_{\omega'}} && (46) \\ \sqrt{n_{\omega'}} \left[n_\omega, e_{\omega'}^\dagger\right] &= 2\pi \sqrt{n_{\omega'}} e_{\omega'}^\dagger \delta(\omega - \omega') && (47) \end{aligned}$$

However, the exponential operators e_ω do not commute in the general case with their adjoint operators $e_\omega^\dagger$:

$$\begin{aligned} e_\omega e_\omega^\dagger &= 1 && (48) \\ e_\omega^\dagger e_\omega &= 1 - \alpha_\omega \Pi_\omega && (49) \end{aligned}$$

where Π_ω projects onto vacuum for field components at frequency ω, and α_ω is a function of ω which depends on the specific definition of the phase operator. It follows that the exponential operators are not necessarily unitary

and, hence, that the phase operators are not hermitian. One gets for example $\alpha_\omega = 1$ in the Susskind-Glogower definition[14], and $\alpha_\omega = 0$ in the Pegg-Barnett definition, which thus corresponds to hermitian phase operators[21]. For all definitions, one may nevertheless write

$$\sqrt{n_\omega} e_\omega^\dagger e_\omega = e_\omega^\dagger e_\omega \sqrt{n_\omega} = \sqrt{n_\omega} \tag{50}$$

It follows that simple relations hold for states orthogonal to vacuum, i.e. states such that the probability for having $n_\omega = 0$ vanishes.

We have given definitions of the phase operators for a field having a whole frequency spectrum, and not only for a monomode field. We are thus able to deal with frequency variation of the phase operators and, in particular, to consider the operators δ_ω' obtained by differentiating phases δ_ω versus frequency, according to the Wigner definition of phase-times[22]. A lot of discussions have been devoted to the significance of such a definition, and of its relation with time observables which can be measured by various techniques[23]. Here, we will emphasize that the operators δ_ω' do not commute with number operators and with energy, thus providing quantum phase-times.

Since the exponential operators e_ω commute (see relation (44)), the frequency derivative δ_ω' of the phase may be defined from the frequency derivative e_ω' of the exponential operator:

$$e_\omega' = i\delta_\omega' e_\omega = i e_\omega \delta_\omega' \tag{51}$$

It may be defined as well from the adjoint exponential operators $e_\omega^\dagger$:

$$\left(e_\omega^\dagger\right)' = -i e_\omega^\dagger \left(\delta_\omega'\right)^\dagger = -i \left(\delta_\omega'\right)^\dagger e_\omega^\dagger \tag{52}$$

It follows from relation (48) that the phase derivative δ_ω' is an hermitian operator, even for non-hermitian definitions of the phase δ_ω:

$$\delta_\omega' = -i e_\omega' e_\omega^\dagger = i e_\omega \left(e_\omega^\dagger\right)' = \left(\delta_\omega'\right)^\dagger \tag{53}$$

Using these properties and definitions (40-42), we may now rewrite the definition (39) of m_ω as:

$$\begin{aligned} m_\omega &= \frac{i}{2}\left\{\left(a_\omega'\right)^\dagger a_\omega - a_\omega^\dagger a_\omega'\right\} & (54) \\ m_\omega &= \sqrt{n_\omega}\delta_\omega'\sqrt{n_\omega} & (55) \end{aligned}$$

The quadratic form m_ω is proportional to the density n_ω, but also to the operator δ_ω' which, as we shall see in the next section, has properties of a quantum position in spacetime.

The operators δ_ω' have been defined from phase operators, so that they are expected to have non vanishing commutators with the number operators[12].

The definition of such commutators is affected by the ambiguities already discussed[14]. We may however state them in a rigorous manner by evaluating the commutators between the densities m_ω and $n_{\omega'}$ (for $\omega > 0$ and $\omega' > 0$):

$$[m_\omega, n_{\omega'}] = -2\pi i \delta' (\omega - \omega')\, n_\omega \tag{56}$$

These relations are unambiguously defined in any quantum state and they are consistent with Dirac-like commutators in states orthogonal to the vacuum (states such that $n_\omega \neq 0$):

$$\sqrt{n_\omega}\, [\delta'_\omega, n_{\omega'}]\, \sqrt{n_\omega} = -2\pi i \delta' (\omega - \omega')\, n_\omega \tag{57}$$

To derive this result, we have used relation (55) and the fact that n_ω and $n_{\omega'}$ are commuting variables.

6 Discussion

A comparison between the relations (37) and (38), which describe respectively the effect of a Lorentz boost and of a change of acceleration on the number density, shows that the latter is equivalent to a Doppler shift of the field frequency which depends on the operator δ'_ω. This property appears to be quite close to a quantum expression of the Einstein equivalence principle, provided that δ'_ω plays the role of a position in spacetime, in consistency with the Wigner definition of phase-times[22]. The semiclassical character of the Wigner definition makes its extension to the definition of a quantum operator difficult. We show now that it is however possible to write down rigorous quantum statements with δ'_ω used like a position in spacetime.

To this aim, we evaluate commutation relations between δ'_ω and the energy operator T_0. Multiplying equation (56) by frequency ω' and integrating over ω', we get (see equation (31)):

$$[T_0, m_\omega] = i\hbar n_\omega \tag{58}$$

We may also introduce the integral m of the density m_ω, in the same manner as the total particle number n from the density n_ω:

$$m = \int_0^\infty \frac{d\omega}{2\pi} m_\omega \tag{59}$$

We deduce from the commutator (58):

$$[T_0, m] = i\hbar n \tag{60}$$

We notice that the commutation relations between m and the creation and annihilation operators have a simple form:

$$[m, a_\omega] = i a'_\omega \qquad \left[m, a^\dagger_\omega\right] = i \left(a'_\omega\right)^\dagger \tag{61}$$

We now discuss these relations from the point of view of the quantum definition of positions in spacetime.

We first discuss the spectral relation (58). Since n_ω is invariant in a translation, we deduce from relation (55):

$$\sqrt{n_\omega}\,[T_0, \delta'_\omega]\,\sqrt{n_\omega} = i\hbar n_\omega \tag{62}$$

For states orthogonal to the vacuum state ($n_\omega \neq 0$), this has the form of a canonical commutator between T_0 and δ'_ω, thus defining δ'_ω as a quantum phase-time.

More exactly, T_0 is the energy associated with the light-cone variable u, so that δ'_ω has to be interpreted as a quantum operator U_ω having this variable u as its classical analog. The same manipulations applied to the counterpropagating field component ψ would lead to the definition of a quantum variable V_ω having the light-cone variable v as its classical analog. Combining these two variables, it is therefore possible to define time- and space-like operators:

$$\delta'^{(\varphi)}_\omega \equiv U_\omega = \tau_\omega - \xi_\omega \qquad\qquad \delta'^{(\psi)}_\omega \equiv V_\omega = \tau_\omega + \xi_\omega \tag{63}$$

which are conjugated to the field energy and momentum:

$$[E, \tau_\omega] = i\hbar \qquad\qquad [P, \xi_\omega] = -i\hbar \tag{64}$$

defined through:

$$E = T_0^{(\varphi)} + T_0^{(\psi)} \qquad\qquad P = T_0^{(\varphi)} - T_0^{(\psi)} \tag{65}$$

This provides quantum definitions of time and space operators τ_ω and ξ_ω, defined at each frequency ω like the semiclassical Wigner definitions.

In order to give a more explicit realisation of quantum positions in spacetime, we now consider the integrated relation (60), in the particular case of a 1-particle state. As already discussed, the notion of a number state is preserved in conformal transformations to accelerated frames; precisely the total particle number n is preserved. In particular, the definition of a 1-particle state ($n = 1$) is the same for accelerated and inertial observers. For such a state, the commutator (60) now reads as a canonical commutator between the energy T_0 and the operator m:

$$[T_0, m] = i\hbar \qquad\qquad n = 1 \tag{66}$$

This relation may be considered as associating a quantum position to the 1-particle state, precisely one position for each light-cone variable. Following the same path as from equation (63) to equation (65), we may then obtain time and space operators τ and ξ associated with the state.

In fact, the operator m is a generalization for quantum fields of the Newton-Wigner quantum position[11]. This position, initially defined for

a wavefunction, is here extended to 1-particle field states. To make this point explicit, we represent each 1-particle state by a function f of frequency or of position:

$$\mid f\rangle = \int_0^\infty \frac{d\omega}{2\pi} f\left[\omega\right] \mid \omega\rangle = \int_{-\infty}^\infty du\ f\left(u\right) \mid u\rangle \tag{67}$$

where we have used Dirac-like ket notations for the basis states:

$$\begin{aligned} \mid \omega\rangle &= a_\omega^\dagger \mid \mathrm{vac}\rangle && (68)\\ \mid u\rangle &= \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega u} \mid \omega\rangle && (69)\end{aligned}$$

Equation (61) thus means that the operator m may be represented in the space of functions f either as the differential operator $(-i\partial_\omega)$ in the frequency domain, or as the multiplication by u in the position domain:

$$m \mid f\rangle = -i \int_0^\infty \frac{d\omega}{2\pi} f'\left[\omega\right] \mid \omega\rangle = \int_{-\infty}^\infty du\ u\ f\left(u\right) \mid u\rangle \tag{70}$$

Its spectral density m_ω can be shown to be related to its symmetrised product with particle number density n_ω:

$$\frac{1}{2}\left\{m, n_\omega\right\} \equiv \frac{1}{2}\left(m n_\omega + n_\omega m\right) = m_\omega + :m n_\omega: \tag{71}$$

where $:\quad:$ denotes normal ordering:

$$:m n_\omega := \frac{i}{2}\int_0^\infty \frac{d\omega'}{2\pi}\left\{\left(a'_{\omega'}\right)^\dagger a_\omega^\dagger a_{\omega'} a_\omega - a_{\omega'}^\dagger a_\omega^\dagger a'_{\omega'} a_\omega\right\} \tag{72}$$

This normal product vanishes when applied to 1-particle field states, so that for such states the density m_ω can be identified, as an operator, with the symmetrised product of position m and particle number density n_ω. Using the commutation relations (61), it can then be rewritten under the form (55) with position m substituted for δ'_ω:

$$m_\omega = \frac{1}{2}\left\{m, n_\omega\right\} = \sqrt{n_\omega} m \sqrt{n_\omega} \qquad\qquad n = 1 \tag{73}$$

Finally, transformations of particle number density to inertial or accelerated frames take the simple form of Doppler shifts of the frequency ((37) and (38)):

$$\begin{aligned} \frac{1}{i\hbar}\left[T_1, n_\omega\right] &= \partial_\omega\left\{\omega n_\omega\right\} && (74)\\ \frac{1}{i\hbar}\left[T_2, n_\omega\right] &= 2\partial_\omega\left\{\omega\sqrt{n_\omega}\delta'_\omega\sqrt{n_\omega}\right\} && (75)\end{aligned}$$

For 1-particle field states, the last relation can also be written:

$$\frac{1}{i\hbar}\left[T_2, n_\omega\right] = 2\partial_\omega \left\{\omega\sqrt{n_\omega} m \sqrt{n_\omega}\right\} \qquad n = 1 \tag{76}$$

This Doppler shift is proportional to the acceleration and to the Newton-Wigner position of the particle.

We may now summarize the results obtained in this paper. In order to take advantage of the conformal symmetry of massless field theories, we have represented accelerated frames by conformal transformations. Invariance of vacuum under such transformations was already known[10]. We have demonstrated that total particle number was also invariant, thus proving the consistency of a point of view where vacuum and number states are the same for inertial and accelerated observers. In contrast with the common Rindler representation of accelerated frames discussed in the introduction, this point of view allows to discuss the effect of acceleration on quantum fields in terms of a redistribution of particle in the frequency domain. Analyzing the transformation of spectral density of particle number from inertial to accelerated frames, we have shown that it may be interpreted in terms of Doppler shifts depending upon position in spacetime, in conformity with the Einstein equivalence principle. This position is defined as the frequency derivative of some phase operators, in analogy with the Wigner definition of phase-times[22]. In the particular case of 1-particle states, it is a generalization to Quantum Field Theory of the Newton-Wigner position operator initially defined for wavefunctions[11]. Considered as a whole, these results constitute a step forward in the direction of a consistent interpretation of the Einstein equivalence principle in the quantum domain.

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